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On the number of contractible triples in 3-connected graphs

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Abstract

McCuaig and Ota proved that every 3-connected graph G on at least 9 vertices admits a *contractible triple*, i.e. a connected subgraph H on three vertices such that $G - V(H)$ is 2-connected. Here we show that every 3-connected graph G on at least 9 vertices has more than $|V(G)|/10$ many contractible triples. If, moreover, G is cubic, then there are at least $|V(G)|/3$ many contractible triples, which is best possible.

Keywords: Connectivity; Contractible edge; Contractible triple

1. Introduction

All *graphs* considered here are supposed to be finite, simple, and undirected. For terminology not defined here the reader is referred to [2] or [3]. For a graph G and a subset X of its vertex set $V(G)$ we write $G(X)$ for the subgraph induced by X in G and $E_G(X)$ for the set of edges connecting a vertex from X to one of $V(G) - X$.

A connected subgraph H of a 3-connected graph G is called *contractible* if $G - V(H)$ is 2-connected, or, equivalently, if the graph $G/V(H)$ obtained from $G - V(H)$ by adding a new vertex and making it adjacent to all neighbors of $V(H)$ in G is 3-connected. A *contractible triple* is a contractible subgraph on three vertices, and an edge xy of G is called *contractible* if $G(\{x, y\})$ is contractible. Tutte proved that every 3-connected graph G on at least 5 vertices contains a contractible edge [8]. It follows already from his proof that G has *more* than one

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contractible edge, and later it has been proven that there must be at least $|V(G)|/2$ many [1], which is best possible in general.

As a generalization of TUTTE's theorem, MCCUAIG and OTA conjectured that for every integer $\ell \geqslant 3$, there exists a (smallest) integer $f(\ell)$ such that every 3-connected graph on at least $f(\ell)$ vertices admits a contractible subgraph on exactly ℓ vertices [7]. Observing that a *cube* $K_2 \times K_2 \times K_2$ has no contractible triples at all, they determined $f(3) = 9$ by showing the following:

Theorem 1. *(See [7].) Every* 3*-connected graph on at least* 9 *vertices has a contractible triple.*

Later, it has been proven that $f(4) = 8$ [6], but the existence of $f(\ell)$ is not settled for any $\ell \geqslant 5$ yet.

Here we concentrate on generalizing Theorem 1 by showing that every 3-connected graph G on at least 9 vertices has more than $|V(G)|/10$ many contractible triples (Theorem 5). This improves to $|V(G)|/3$ for cubic graphs G (Corollary 1). As the contractible triples of some 3-connected cubic graph in which every vertex is on exactly one triangle are precisely these triangles, the bound in Corollary 1 is sharp, and the order of the bound in Theorem 5 in terms of $|V(G)|$ is best possible.

2. Links, extendibility, expanded wheels

Let us recall some concepts from [6]. A *link* L in some graph G is an induced subpath of G such that each vertex of L has degree 2 in G. It is called *maximal*, if there is no link M in G such that L is a proper subgraph of M, and it is called *removable* if $G - V(L)$ is 2-connected. Hence every removable link in a 2-connected graph is maximal. We call two disjoint subgraphs P, Q of G *nonadjacent* if $V(P) \cap N_G(V(Q)) = \emptyset$.

A contractible subgraph H of some 3-connected graph G is called *extendible* if $G(V(H) \cup \{z\})$ is contractible for some $z \in V(G) - V(H)$. If there is only one such z then we call H *uniquely extendible*. A contractible edge xy is called *extendible* if $G(\{x, y\})$ is extendible. Extendability and the presence of removable links in $G - V(H)$ are intimately connected by the following theorems.

Theorem 2. *(See [6].) If a contractible subgraph H of some* 3*-connected graph G is not extendible then $G - V(H)$ either induces a cycle or admits two disjoint nonadjacent removable links each of which is of order at least* 2.

Theorem 2 extends easily to the case of uniquely extendible subgraphs, as it has been discussed in [5]. We need the statement only for $|V(H)| = 1$:

Theorem 3. *(See [5, Theorem 12].) If a vertex h of some* 3*-connected graph G is incident with exactly one contractible edge then $G - h$ admits a removable link of order at least* 2.

When looking for contractible subgraphs in some graph G we often may assume that G is minimally 3-connected, as every contractible subgraph of G is a contractible subgraph of every supergraph of G on the same vertex set. This has several advantages; we extract two of them from the considerations in [4].

Lemma 1. *(See [4, (9)].) Every triangle of a minimally* 3*-connected graph has at least two vertices of degree* 3*.*

Lemma 2. *(See [4, Satz 6].) Every minimally* 3*-connected graph G has at least $\frac{2}{5}(|V(G)|+3)$ vertices of degree* 3*.*

Let us first count the number of contractible triples in a very special class of minimally 3-connected graphs (those in which there is an edge whose "contraction to h" produces a wheel with center h).

Lemma 3. *Let G be a minimally* 3*-connected graph on at least* 6 *vertices and let xy be a contractible edge such that $G-\{x,y\}$ is a cycle. Then G has at least $|V(G)|-2-|\{z\in\{x,y\}\colon d_G(x)\leqslant 4\}|$ many contractible triples.*

Proof. Set $C := G-\{x,y\}$. If there exists a $z\in N_G(x)\cap N_G(y)$ then $z\in V(C)$ and $d_G(z)=4$, so $d_G(x)=d_G(y)=3$ by Lemma 1, implying that $|V(G)|=|V(C)|+|\{x,y\}|=|(N_G(x)\cup N_G(y))\cap V(C)|+2\leqslant 3+2$—a contradiction. Hence $N_G(x)\cap V(C)$ and $N_G(y)\cap V(C)$ form a partition of $V(C)$.

Let $\mathcal{Q}$ denote the set of subpaths of C on three vertices. For $z\in\{x,y\}$, set $\mathcal{Q}_z := \{P\in\mathcal{Q}\colon N_G(z)\cap V(C)\subseteq V(P)\}$ and observe that $P\in\mathcal{Q}$ is contractible if and only if $P\notin\mathcal{Q}_x\cup\mathcal{Q}_y$. Furthermore, if $N_G(z)\cap V(C)$ consists of two adjacent vertices c,d then let $\mathcal{R}_z$ consist of the triangle cdz, which is contractible, otherwise set $\mathcal{R}_z:=\emptyset$.

If $d_G(z)\geqslant 5$ then $\mathcal{Q}_z=\emptyset$, and if $d_G(z)=4$ then $|\mathcal{Q}_z|\leqslant 1$. If $d_G(z)=3$ then $|\mathcal{Q}_z|\leqslant 2$, where equality is attained if and only if $\mathcal{R}_z\neq\emptyset$. Hence $|\mathcal{Q}_z|\leqslant\varepsilon_z+|\mathcal{R}_z|$, where $\varepsilon_z:=1$ if $d_G(z)\leqslant 4$ and $\varepsilon_z:=0$ otherwise.

Now $(\mathcal{Q}-(\mathcal{Q}_x\cup\mathcal{Q}_y))\cup\mathcal{R}_x\cup\mathcal{R}_y$ consists of $|V(C)|-|\mathcal{Q}_x|-|\mathcal{Q}_y|+|\mathcal{R}_x|+|\mathcal{R}_y|\geqslant|V(C)|-\varepsilon_x-\varepsilon_y=|V(G)|-2-|\{z\in\{x,y\}\colon d_G(x)\leqslant 4\}|$ many contractible triples, which proves the lemma. □

3. Cube fragments as certificates for not being on contractible triples

Let $T\subseteq V(G)$ be an arbitrary separating set of G. A *T-fragment* is the union of the vertex sets of at least one but not of all components of $G-T$. If F is a T-fragment then so is $\overline{F}^{(T,G)} := V(G)-(F\cup T)$, where we omit the superscript (T,G) if it is clear from the context.

If F' is a T'-fragment and $F\cap F'\neq\emptyset$ then $F\cap F'$ is a $(T-\overline{F'})\cup(T'-\overline{F})$-fragment, a fact which will be used throughout without any further reference.

A vertex $x\in T$ is *essential* for T if $T-x$ does not separate G, or, equivalently, if x has neighbors in every component of $G-T$. In particular, if all but at most one neighbor of x are contained in T then x cannot be essential for T.

Observe that

$$(*)\qquad \begin{cases}\text{if } T, T' \text{ are separators and } T \text{ separates two essential vertices of } T' \text{ from each other}\\ \text{then } T' \text{ separates } T\text{, too:}\end{cases}$$

For let F be a T-fragment and both $x\in F$ and $y\in\overline{F}$ be essential members of T'; then for every T'-fragment F' there exists an x,y-path of length at least 2 whose inner vertices are in F'; so each T'-fragment must intersect T, and, consequently, T' separates T.

Let $\kappa(G)$ denote the (*vertex*) *connectivity* of G, and let $\mathcal{T}(G)$ denote the set of *smallest separating sets* of G, i.e. the separating sets of cardinality $\kappa(G)$. It is obvious that every member of some $T \in \mathcal{T}(G)$ is essential for that T. Moreover, it is easy to see that an edge xy of a 3-connected graph nonisomorphic to K_4 is contractible if and only if $\{x, y\}$ is not a subset of any smallest separating set.

A set F of vertices of degree 3 in a graph G is a *cube fragment* of G if the graph obtained from $G(F \cup N_G(F))$ by adding a new vertex and making it adjacent to all vertices of $N_G(F)$ is a cube. In this case, F contains exactly one vertex x not adjacent to $N_G(F)$, which is called its *peak*. Obviously, the peak of a cube fragment of a 3-connected graph is not contained in any contractible triple. The main result of this section states that if, conversely, x is not on a contractible triple but on a contractible edge xy where $d_G(x) = d_G(y) = 3$, then it must be the peak of a particular cube fragment, unless G is one of some small exceptional graphs.

Let $W_4 = C_4 * K_1$ denote the wheel on 5 vertices.

Theorem 4. *Let xy be a contractible edge in a 3-connected graph G nonisomorphic to one of W_4, $K_2 \times K_3$, $K_{3,3}$ such that $d_G(x) = d_G(y) = 3$ and such that x is not contained in a contractible triple. Then x is the peak of a cube fragment F and all vertices in $N_G(F)$ have degree 3 in G.*

Proof. Clearly, xy is not extendible and $E_G(\{x, y\}) = 4$. It is easy to see that if $G - \{x, y\}$ induces a 3- or 4-cycle then $G \cong W_4$, $G \cong K_3 \times K_2$, or $G \cong K_{3,3}$.

Hence, by Theorem 2, $G - \{x, y\}$ admits a pair $P = pq$, $S = st$ of nonadjacent removable links of order 2, where each of p, q, s, t has degree 3 in G. If $V(P) \subseteq N_G(y)$ or $V(S) \subseteq N_G(y)$ then stx or pqx would be a contractible triangle. Hence we may assume without loss of generality that $px, qy, sx, ty \in E(G)$. Let $X := \{p, q, s, t, x, y\}$, and let a, b, c, d denote the neighbors of p, q, s, t, respectively, in $V(G) - X$. (Some of them may coincide.)

If $|N_G(X)| = 2$ then $\{a, b\} = \{c, d\}$ and $ab \in E(G)$; if $(a, b) = (d, c)$ then $G(\{p, s, x\})$ would be a contractible triple, and if, otherwise, $(a, b) = (c, d)$ then G would be a cube, in which *every* vertex is the peak of some appropriate cube fragment.

Hence we may assume that $|N_G(X)| > 2$. Note that px is contractible, since $G - \{p, q, x, y\}$ is 2-connected and q, y are adjacent to each other and to distinct vertices of $G - \{p, q, x, y\}$. Since x is not contained in a contractible triple, px is not extendible. By Theorem 2, $G - \{p, x\}$ has two distinct nonadjacent removable links. As q, y are on the same maximal link of $G - \{p, x\}$, a, s must form another removable link of $G - \{p, x\}$. This implies $a = b$ and $d_G(a) = 3$. Hence $F := \{p, s, x, y\}$ is a cube fragment, x is its peak, and every vertex in $N_G(F) = \{a, q, t\}$ has degree 3. □

Although the peak of a cube fragment is not on a contractible triple, it is often possible to find a number of contractible triples "close" to x as follows:

Lemma 4. *Let F be a cube fragment of a 3-connected graph G nonisomorphic to a cube such that every vertex in $T := N_G(F)$ has degree 3. Then any of the six paths of order 3 which intersect each of $F, T, \overline{F}$ is contractible.*

Proof. Let $x \in T$, let w be the vertex in $N_G(x) \cap \overline{F}$, and let $y \neq z$ be the two vertices in $N_G(t) \cap F$. Then wx is contractible, for otherwise there would be a vertex v such that $\{v, w, x\}$ separates y from z—but there are two openly disjoint y, z-paths in $G(F \cup T - \{x\})$ and thus, in $G - \{w, x\}$, contradiction. Hence for distinct a, b in $\overline{F} - \{w\}$ there exist two openly disjoint

a, b-paths in $G - \{w, x\}$; as at most one of them intersects $F \cup T$ and as $G(F \cup T - \{w, x, y\})$ is connected, there exist two openly disjoint a, b-paths in $G - \{w, x, y\}$, too. Since G is not a cube, the two vertices in $T - \{x\}$ have distinct neighbors in $\overline{F}$, and so for each $c \in (F \cup T) - \{x, y\}$ there exist two $c, \overline{F}$-paths in $G - \{w, x, y\}$ which have only c in common. Hence $G - \{w, x, y\}$ is 2-connected. $\square$

A combination of Theorem 4 and Lemma 4 leads now to a sharp bound for the number of contractible triples in a *cubic* 3-connected graph.

Corollary 1. *Every cubic* 3*-connected graph G on at least* 9 *vertices has* $|V(G)|/3$ *many contractible triples.*

Proof. Consider any $x \in V(G)$. If x is contained in exactly one 4-cycle C of G then let $f(x)$ denote the vertex in C not adjacent to x, if x is contained in exactly two 4-cycles and these cycles share exactly one vertex y distinct from x then let $f(x) := y$, and in all other cases, let $f(x) := x$.

Let $\mathcal{F}$ be the set of all cube fragments F in G. For each $F \in \mathcal{F}$, let $A(F) := F \cup N_G(F)$ and observe that, since G is not a cube, $E_G(A(F))$ consists of 3 independent edges and $N_G(A(F))$ consists of 3 vertices. Consequently, if $x \in A(F)$ then $f(x)$ is the peak of F, and so $A(F)$, $A(F')$ are disjoint for distinct F, F' from $\mathcal{F}$.

For $F \in \mathcal{F}$, let $B(F) := A(F) \cup N_G(A(F))$; so $|B(F)| = 7 + 3 = 10$.

Consider $x \in V(G)$. If $x \in B := \bigcup_{F \in \mathcal{F}} B(F)$ then choose any $\varphi(x) \in \mathcal{F}$ with $x \in B(\varphi(x))$ and define $\alpha(x)$ to be the set of those six paths of order 3 which intersect each of $\varphi(x), N_G(\varphi(x)), \overline{\varphi(x)}$. By Lemma 4, the paths in $\alpha(x)$ are contractible.

If, otherwise, $x \notin B$, then x must be on a contractible edge of G (for otherwise, x had at least four neighbors by Theorem 2, applied to $G(\{x\})$ for H—but G is cubic). Hence x must be on some contractible triple H by Theorem 4, and we set $\beta(x) := H$.

If $\varphi(x) \neq \varphi(x')$ for $x, x' \in B$ then $\alpha(x), \alpha(x')$ are disjoint, as a path in $\alpha(x)$ must intersect $\varphi(x)$, a path in $\alpha(x')$ must intersect $\varphi(x')$, and $\varphi(x), \varphi(x')$ are disjoint.

If $x \in V(G) - B$ and $x' \in B$ then $\beta(x) \notin \alpha(x')$ as the vertices of every path of $\alpha(x')$ are contained in $B(\varphi(x'))$, whereas $x \in V(\beta(x))$ is not.

Since $|\varphi^{-1}(F)| \leqslant |B(F)| = 10$ for all $F \in \mathcal{F}$ and $|\beta^{-1}(H)| \leqslant 3$ for every contractible subgraph H, we deduce that there are at least $|V(G) - B|/3 + 6 \cdot |F| \geqslant |V(G) - B|/3 + 6 \cdot |B|/10 \geqslant |V(G)|/3$ many contractible triples. $\square$

4. The general argument

Unfortunately, the statement of Theorem 4 does not generalize in a simple way when there is no restriction to $d_G(y)$. To illustrate the problems let us have a look at the central vertex y in the graph of Fig. 1. Its neighbor x in the north is not on any contractible triple. Suppose we wanted to assign just one contractible triple $\gamma(x)$ to x, similarly as we did with the six paths of $\alpha(x)$ in the proof of Corollary 1. Theorem 4 does not apply here, but, by Theorem 2, we still find a contractible edge xy_x incident with x; in our example, $y_x := y$ would do it. Since xy is not extendible, it is then possible to employ Theorem 2 once more to find a contractible triple $\gamma(x)$ which either contains y or is in the neighborhood of $\{x, y\}$ (and we will do this later in the proof of Theorem 5). The problem is that y could have *many* other neighbors x' of degree 3 not on a contractible triple such that $x'y$ is contractible—in Fig. 1 half of the edges $x'y$ play the same

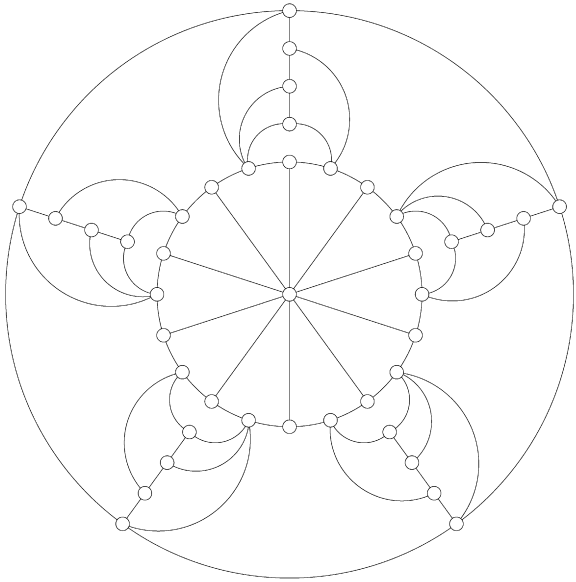

Fig. 1. When Theorem 4 is not applicable.

role—and to each of them one and the same contractible triple could have been assigned. Hence $\gamma(x)$ is possibly "far from being an injection" and useless to bound the number of contractible triples from below.

We will overcome this problem by being more careful when choosing y_x. The following lemma is the key observation in our counting argument.

Lemma 5. *Let G be a minimally 3-connected graph nonisomorphic to K_4. Let*

$$W := \{x \in V(G)\colon\ d_G(x) = 3,\ x \text{ is not on a contractible triple}\},$$

and for $y \in V(G)$, let

$$X(y) := \{x \in N_G(y) \cap W\colon\ xy \text{ is contractible}\}.$$

Then for every $x \in W$ there exists a $y \in V(G)$ such that $x \in X(y)$ and $d_G(y) = 3$ or $X(y) = \{x\}$.

Proof. Note that $\kappa(G) = 3$ and let $x \in W$. The subgraph induced by x is contractible, and it is extendible by Theorem 2 since $G \ncong K_4$. Hence xy is contractible for some $y \in N_G(x)$, that is, $x \in X(y)$. Let $a \neq b$ be the two vertices in $N_G(x) - \{y\}$. We may assume that $d_G(y) > 3$ and that there exists an $x' \in X(y) - \{x\}$ (for otherwise the statement would follow). Since $G - \{x, y\}$ is 2-connected and G is minimally 3-connected, $a, b \notin N_G(y)$ by Lemma 1.

Since $G(\{x, y, x'\})$ is not contractible, there exists a vertex t such that $T := \{x, y, x', t\}$ separates G. Since xy is contractible, $|T| = 4$, and since $x'y$ is contractible, x is essential for T. Hence there exists a T-fragment F_a such that $a \in F_a$ and $b \in \overline{F_a} =: F_b$; in particular, $ab \notin E(G)$.

It follows from Theorem 3 that one of xa, xb is contractible, so $G - \{x, a, b\}$ is connected, and it possesses a cut vertex as $G(\{x, a, b\})$ is not contractible. We choose a cut vertex z in $G - \{x, a, b\}$ and, *if possible, we choose z nonadjacent to t.*

Claim 1. *$z \neq y$.*

Suppose, to the contrary, that $z = y$. Then a, b are essential for $T_0^+ := \{x, a, b, z = y\}$ (as xy is contractible), so T_0^+ separates x' from t (cf. $(*)$). Since $N_G(x) \subseteq T_0^+$, we find a $T_0 := \{a, b, z = y\}$-fragment F_0 such that $t \in F_0$ and $x, x' \in \overline{F_0}$. Then $F_a \cap \overline{F_0} = \emptyset$ and $F_b \cap \overline{F_0} = \emptyset$, for otherwise one of the latter sets would be an $\{a, x', z = y\}$-fragment or a $\{b, x', z = y\}$-fragment, respectively, as they contain no neighbors of x—but this violates the contractibility of $x'y$. Consequently, $\overline{F_0} = \{x, x'\}$, and $N_G(x') = N_G(x)$.

Let $c \in \{a, b\}$. Assume for a while that $d_G(c) = 3$. Then we may suppose that cx is not contractible (for otherwise the statement would follow for $y := c$), hence c, x are contained in some $T_1 \in \mathcal{T}(G)$. As x is essential for T_1, T_1 separates y from the vertex d in $\{a, b\} - \{c\}$, and as x' is a common neighbor of y, d, $x' \in T_1$ follows. But then a has only one neighbor outside T_1, so it cannot be essential for T_1, which is absurd.

It follows that $d_G(a), d_G(b) > 3$. As $ay, by \notin E(G)$, the sets $L := F_a \cap F_0$ and $R := F_b \cap F_0$ are both nonempty. By Theorem 3, one of ax, bx is contractible; without loss of generality, let bx be contractible. Since $G(\{x, x', b\})$ is not contractible, there exists a vertex v such that $T_2 := \{x, x', b, v\}$ separates G, and since xb is contractible, $|T_2| = 4$ follows, and x' is essential for T_2. Consequently, there exists a T_2-fragment F_2 such that $a \in F_2$ and $z = y \in \overline{F_2}$. As L is an $\{a, t, z = y\}$-fragment, there exists an $a, z = y$-path of length at least 2 whose inner vertices are in $L \neq \emptyset$, so $v \in L$ follows.

But then all vertices in $R \cup \{t, z = y\}$ are in the same component of $G - T_2$ (and, thus, in $\overline{F_2}$), as R is a $\{b, t, z = y\}$-fragment and for each vertex $r \in R \neq \emptyset$ there exists a system of three $r, \{b, t, z = y\}$-paths which have pairwise only r in common and whose inner vertices are in R. In particular, $t \in \overline{F_2}$. For each $\ell \in L - \{v\}$, there exists a system of three $\ell, \{a, t, z = y\}$-paths which have pairwise only ℓ in common and whose inner vertices are in L; either the ℓ, t- or the $\ell, z = y$-path avoids v, so $\ell \in \overline{F_2}$. Consequently, $F_2 = \{a\}$, and $N_G(a) = \{x, x', v\}$, contradicting $d_G(a) > 3$.

This proves Claim 1.

Since x is not essential for the separator $\{x, a, b, z\}$ of G, $T_0 := \{a, b, z\}$ is in $\mathcal{T}(G)$, and we may take a T_0-fragment F_0 such that $x \in \overline{F_0}$. By Claim 1, it follows $y \in \overline{F_0}$ and, thus, $F_0 \cap (\{a, b\} \cup N_G(y)) = \emptyset$.

Claim 2. *$t \in F_0$ and $x' \in \overline{F_0}$.*

There is an a, b-path P of length at least 2 whose inner vertices are in F_0. Hence T intersects F_0. Since $x, y \in \overline{F_0}$, $x' \in N_G(y) \subseteq (T_0 \cup \overline{F_0}) - \{a, b\}$ and $t \in F_0$ follow; assume, to the contrary, that $x' \in T_0$, so $x' = z$; for some $c \in \{a, b\}$, F_c contains a neighbor of y, so $L := F_c \cap \overline{F_0}$ is nonempty; since x has no neighbor in L, L must be a $\{c, x', y\}$-fragment, contradicting the contractibility of $x'y$. This proves Claim 2.

Claim 3. *Let $c \in \{a, b\}$. If xc is contractible then the edges cf with $f \in F_0$ are not.*

We may assume $c = b$ without loss of generality. Let $f \in N_G(b) \cap F_0$ and suppose, to the contrary, that bf is contractible. Since $G(\{x, b, f\})$ is not contractible, there exists a vertex v such that $T_1 := \{x, b, f, v\}$ separates G. Since bx is contractible, $|T_1| = 4$, and since xb, bf are contractible, x, f (and v) are essential for T_1. Hence T_1 separates T_0 (cf. $(*)$), and, therefore, it separates a from z. Let F_1 be a T_1-fragment with $a \in F_1$ and $z \in \overline{F_1}$. As x is essential for T_1, $y \in \overline{F_1}$ follows.

If $v \in F_0$ then $\overline{F_0} \cap \overline{F_1}$ is a $\{b, x, z\}$-fragment, contradicting the contractibility of bx.

Consequently, $v \in \overline{F_0}$, and $F_0 \cap F_1 = \emptyset$ (for otherwise $F_0 \cap F_1$ would be an $\{a, b, f\}$-fragment, contradicting the contractibility of bf). Similarly, $F_0 \cap \overline{F_1} = \emptyset$, for otherwise the latter set would be a $\{b, f, z\}$-fragment, contradicting the contractibility of bf. Hence $F_0 = \{f\}$.

If $L := \overline{F_0} \cap F_1$ was empty then $F_1 = \{a\}$ would follow, and $N_G(a) = \{f, v, x\}$. We may assume that xa is not contractible, for otherwise our statement would follow with $y := a$. Hence there exists a vertex z' such that $T_0' := \{x, a, z'\} \in \mathcal{T}(G)$. There exists a T_0'-fragment F_0' with $b \in F_0'$ and $y \in \overline{F_0'}$. It follows $f \in T_0' \cup F_0'$, so $f \in F_0'$ and $v \in \overline{F_0'}$ as a is essential for T_0'.

Now z' is a cut vertex of $G - \{a, b, x\}$ (separating v from f). *By choice of* z *we conclude* $z' = z$.

If $R := F_0' \cap \overline{F_0}$ was nonempty then R would be an $\{a, b, z, x\}$-fragment; but neither a nor x have neighbors in R, so R is a $\{b, z\}$-fragment—contradiction. Hence $F_0' = \{f, b\}$. But then $d_G(b) = 3$ as b is not adjacent to a, and the statement of our lemma follows for $y := b$.

Consequently, $L \neq \emptyset$, and, as x has no neighbor in L, L is an $\{a, b, v\}$-fragment. Therefore, we find an a, b-path P of length at least 2 whose inner vertices are in L. P, afb are two openly disjoint a, b-paths which neither contain y nor its neighbors x, x', because y is in $\overline{F_0} \cap \overline{F_1}$; but then T cannot separate a from b in G, a contradiction.

This proves Claim 3.

Claim 4. *For* $c \in \{a, b\}$, *either* $z \in F_c$, *or* $F_c = \{c\}$ *and* $N_G(c) = \{x, x', t\}$.

Suppose that $z \in \overline{F_c}$. Then $F_c \cap F_0 = \emptyset$, as otherwise the latter set would be a $\{c, t\}$-fragment. Furthermore, $F_c \cap \overline{F_0} = \emptyset$, for otherwise the latter set would be an $\{x, y, x', c\}$-fragment without neighbors of x and, therefore, a $\{y, x', c\}$-fragment, which contradicts the contractibility of $x'y$. Hence $F_c \subseteq T_0$, so $F_c = \{c\}$. Since $yc \notin E(G)$, $N_G(c) = T - \{y\} = \{x, x', t\}$. This proves Claim 4.

As we noticed before, by Theorem 3, there exists a $c \in \{a, b\}$ such that cx is contractible. We may assume that $d_G(c) > 3$ (for otherwise the statement would follow with $y := c$). By Claim 4, $z \in F_c$, and, again by Claim 4, $\overline{F_c}$ consists of the vertex $d \in \{a, b\} - \{c\}$, where $N_G(d) = \{x, x', t\}$. We may assume that xd is not contractible, for otherwise the statement of our lemma would follow for $y := d$.

Claim 5. $x'c \notin E(G)$.

Suppose that $x'c \in E(G)$. Since xd is not contractible, there exists a vertex v such that $T_2 := \{x, d, v\}$ separates G. As x is essential for T_2, there exists a T_2-fragment F_2 such that $c \in F_2$ and $y \in \overline{F_2}$. Since x' is a common neighbor of c, y, $x' \in T_2$ follows, implying that $x' = v$. But then d is not essential for T_2, a contradiction—which proves Claim 5.

Claim 6. *For* $f \in N_G(c) \cap (\overline{F_0} - \{x, x'\})$, cf *is not contractible.*

Suppose, to the contrary, that cf is contractible. Since $G(\{x, c, f\})$ is not contractible, there exists a vertex v such that $T_2 := \{x, c, f, v\}$ separates G. Since cx is contractible, $|T_2| = 4$, and since cf is contractible, x is essential for T_2. Hence there exists a T_2-fragment F_2 such that

$d \in F_2$ and $y \in \overline{F_2}$. As x' is a common neighbor of d, y, $x' \in T_2$ follows, implying that $x' = v$. Let p be the neighbor of x' distinct from d, y.

If $p \in \overline{F_2}$ then d is the unique vertex in $N_G(\{x, x'\}) \cap F_2$, and $(T_2 - \{x, x'\}) \cup \{d\} = \{c, f, d\}$ separates G (it separates $t \in F_2 \neq \{d\}$ from y), which contradicts the contractibility of cf. If, otherwise, $p \in F_2$ then y is the unique vertex in $N_G(\{x, x'\}) \cap \overline{F_2}$. Since $d_G(y) > 3$ and $c \notin N_G(y)$, $\overline{F_2} \neq \{y\}$, and so $(T_2 - \{x, x'\}) \cup \{y\}$ separates G, which contradicts again the contractibility of cf.

This proves Claim 6.

We are now able to accomplish the proof by showing $X(c) = \{c\}$. It suffices to prove that for every $f \in N_G(c) - \{x\}$, cf is not contractible. This is immediate if $f = z \in T_0$, it follows from Claim 3 if $f \in F_0$, and it follows from Claim 5 and Claim 6 if $f \in \overline{F_0}$. $\square$

Theorem 5. *Every* 3*-connected graph G on at least* 9 *vertices has more than* $|V(G)|/10$ *contractible triples.*

Proof. Let $\mathcal{F}$ be the set of all cube fragments F in G such that all vertices in $N_G(F)$ have degree 3 in G, and let $A(\cdot)$, $B(\cdot)$, B be as in the proof of Corollary 1. Observe that $A(F) \cap A(F') = \emptyset$ for $F \neq F'$ in $\mathcal{F}$ and $|B(F)| = 10$ for all $F \in \mathcal{F}$ hold under our present, weaker conditions, too. Let V_3 denote the set of vertices of degree 3 in G.

Consider $x \in V_3$. If $x \in B$ then we choose any $\varphi(x) \in \mathcal{F}$ with $x \in B(\varphi(x))$ and define $\alpha(x)$ to be the set of those six paths of order 3 which intersect each of $\varphi(x), N_G(\varphi(x)), \overline{\varphi(x)}$. By Lemma 4, the paths in $\alpha(x)$ are contractible.

If $x \in V_3 - B$ is on a contractible triple H then we set $\beta(x) := H$.

If $x \in V_3 - B$ is not on a contractible triple then, by Lemma 5, there exists a vertex $y := y_x$ such that xy is contractible and either $d_G(y) = 3$, or $d_G(y) > 3$ and there is no $x' \in N_G(y)$ of degree 3 not on a contractible triple such that $x'y$ is contractible. Note that if $d_G(y) = 3$ then x would be the peak of some cube fragment $F \in \mathcal{F}$ by Theorem 4 and, thus, in B. Hence $d_G(y) > 3$.

If $G' := G - \{x, y\}$ is a cycle then the entire statement of the theorem follows from Lemma 3. Otherwise, G' contains a pair of disjoint nonadjacent removable links P, Q with $|V(Q)| \geqslant |V(P)| \geqslant 2$ by Theorem 2. Since G is minimally 3-connected, every vertex in $V(P) \cup V(Q)$ must be adjacent to exactly one of x, y by Lemma 1. We now define a contractible triple $\gamma(x) := H_{xy}$ as follows.

If $P = pq$ then x cannot be adjacent to $V(P)$ (for otherwise $G(\{p, q, x\})$ would be a contractible triple as $d_G(y) > 3$), and we define $\gamma(x)$ to be the contractible triangle $H_{xy} := G(\{p, q, y\})$. Otherwise, $|V(Q)| \geqslant |V(P)| \geqslant 3$. If P or Q contains a subpath pqr of order 3 such that $p, q, r \in N_G(y)$ then this path is contractible and we set $\gamma(x) := H_{xy} := pqr$. Otherwise, x has a neighbor p in $V(P)$ and a neighbor $q \in V(Q)$. If p was an inner vertex of P and q was an inner vertex of Q then pxq would be a contractible triple, which is not possible; therefore, there are adjacent neighbors v, w of y in P or in Q, and we choose $\gamma(x) := H_{xy} = vwy$, which is a contractible triangle.

Let $\mathcal{C}$ be the set of contractible triples and let $C := \bigcup_{C \in \mathcal{C}} V(C)$. Then

$$\alpha : V_3 \cap B \to \mathcal{P}(\mathcal{C}),$$
$$\beta : (V_3 - B) \cap C \to \mathcal{C}, \quad \text{and}$$
$$\gamma : (V_3 - B) - C \to \mathcal{C}.$$

For $x, x' \in (V_3 - B) - C$ and for distinct $y = y_x$, $y' = y_{x'}$ we observe that $H_{xy} \neq H_{x'y'}$, as y can be reconstructed from H_{xy} to be the unique common neighbor of all vertices of degree 3 in $V(H_{xy})$. Since $y_x \neq y_{x'}$ for $x \neq x'$ by choice of $y_x, y_{x'}$, γ is an injection.

For $x \in (V_3 - B) \cap C$ and $x' \in V_3 \cap B$, $\beta(x)$ is not contained in $\alpha(x')$, since the vertices of every path of $\alpha(x')$ are contained in $B(\varphi(x'))$, whereas $x \in V(\beta(x))$ is not.

For $x \in (V_3 - B) - C$ and $x' \in V_3 \cap B$, $\gamma(x)$ is not contained in $\alpha(x')$, since the two vertices in $A(\varphi(x')) \subseteq V_3$ of any path in $\alpha(x')$ do not have a common neighbor at all, whereas y_x is the common neighbor of the vertices of degree 3 in $\gamma(x)$.

Since $|\varphi^{-1}(F)| \leqslant |B(F)| = 10$ and $|(\beta \cup \gamma)^{-1}(H)| \leqslant |\beta^{-1}(H)| + |\gamma^{-1}(H)| \leqslant |H| + 1 \leqslant 3$ for all $H \in \mathcal{C}$, we thus deduce that there are at least $|V_3 - B|/4 + 6 \cdot |\mathcal{F}| \geqslant |V_3 - B|/4 + 6 \cdot |B|/10 \geqslant |V_3|/4$ many contractible triples in G.

As $|V_3| > \frac{2}{5}|V(G)|$ by Lemma 2, the statement follows. $\square$

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